M5S3 - Interpretation of Confidence Intervals

Professor Jarad Niemi

STAT 226 - Iowa State University

October 17, 2018

Outline

- Interpretation of probability
  - Frequentist
  - Bayesian

Frequentist interpretation of probability

Interpretation

The frequentist interpretation of probability is that probability is the long-run relative frequency of an event.

Thus, if we have a sequence of independent and identically distributed binary random variables $I_1, I_2, \ldots$ where I_i is the indicator of the event occurring in the ith trial, i.e.

$$I_i = \begin{cases} 1 & \text{if the event occurs in the } i\text{th trial} \\ 0 & \text{if the event does not occur in the } i\text{th trial.} \end{cases}$$

Let $S_m = \sum_{i=1}^m I_i$ be the number of events that have occurred in the first m trials. The probability p is defined as

$$p = \lim_{m \to \infty} \frac{S_m}{m}$$

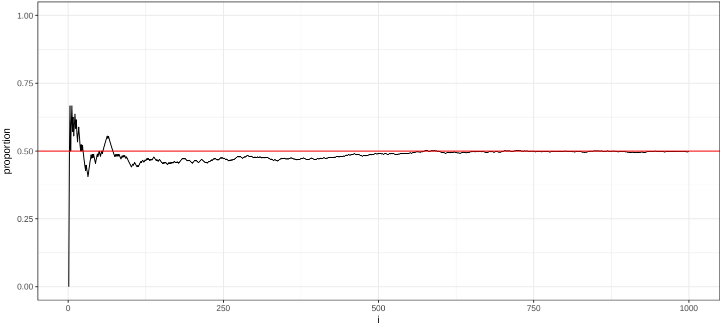
where m is the number of trials

Coin flipping example

Let I_i be the indicator that the ith coin flip is heads, i.e.

$$I_i = \begin{cases} 1 & \text{if he } i\text{th coin flip is heads} \\ 0 & \text{if the } i\text{th coin flip is not heads,} \end{cases}$$

Now we define the probability as the proportion of heads as the number of flips tends to infinity.

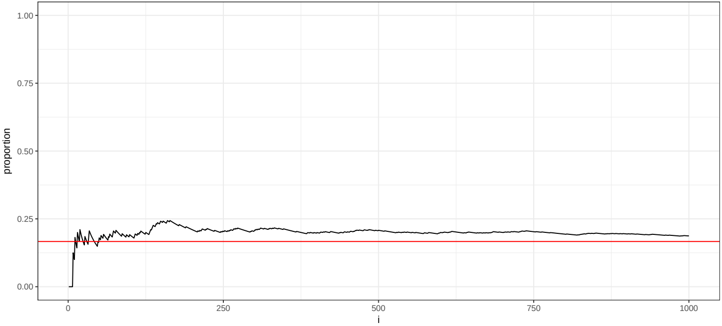


Die rolling example

Let I_i be the indicator of the ith die roll being a 1, i.e.

$$I_i = \begin{cases} 1 & \text{if the } i\text{th die roll is 1} \\ 0 & \text{if the } i\text{th die roll is not 1,} \end{cases}$$

Now we define the probability as the proportion of 1s as the number of rolls tends to infinity.



Construction of confidence intervals

Recall that the formula for a $100(1-\alpha)\%$ confidence interval (CI) based on the standard normal is

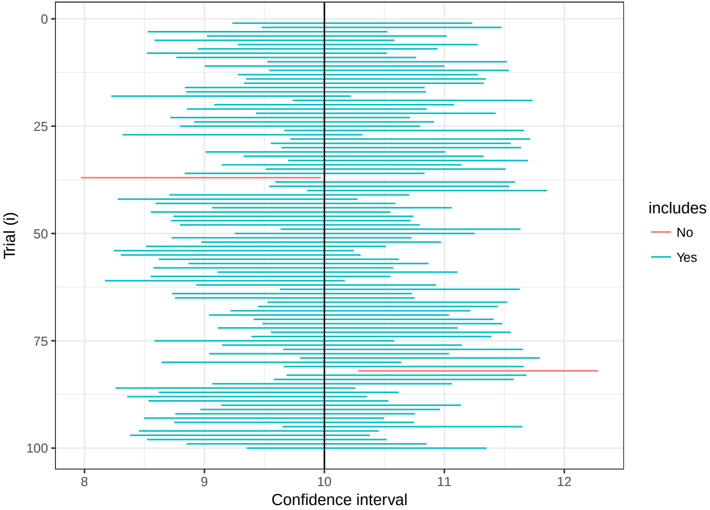
$$\overline{X} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}.$$

We obtained this interval by calculating the following probability

$$P\left(\overline{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \overline{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = 1-\alpha.$$

Thus a confidence interval has random endpoints since $\overline{X}$ is random. We can imagine performing this procedure repeatedly and calculating the proportion of times the CI includes μ.

Let $X_i \stackrel{iid}{\sim} N(10, 1^2)$ with $n = 4$. Then a 95% CI based on the Empirical Rule is $\overline{X} \pm 2 \cdot 1/\sqrt{4} = \overline{X} \pm 1$.



Interpretation of Confidence Intervals

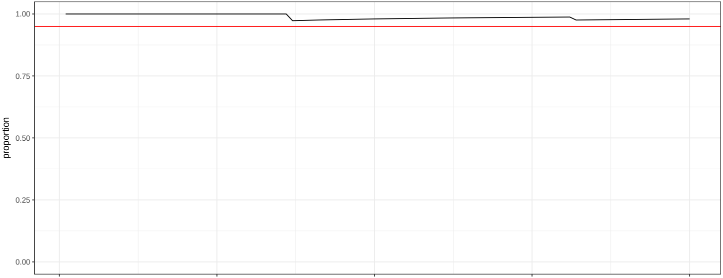
Let I_i be the indicator that the ith $100(1-\alpha)\%$ confidence interval (CI) for the population mean μ contains μ, i.e.

$$I_i = \begin{cases} 1 & \text{if the } i\text{th CI includes } \mu \\ 0 & \text{if the } i\text{th CI does not include } \mu, \end{cases}$$

Then

$$\lim_{m\to\infty} \frac{S_m}{m} = 1 - \alpha.$$

where $S_m = \sum_{i=1}^m I_i$.



Relation to binomial distribution

Recall that a random variable Y has a binomial distribution if

$$Y = \sum_{i=1}^{n} I_i$$

where I_i are independent and identically distributed (iid) Bernoulli random variables with a common probability of success p. Here

$$I_i = \begin{cases} 1 & \text{if CI } i \text{ includes } \mu \\ 0 & \text{if CI } i \text{ does not include } \mu \end{cases}$$

Since each CI has probability $1-\alpha$ of including μ, we have $I_i \stackrel{iid}{\sim} Ber(1-\alpha)$ and $Y \sim Bin(n, 1-\alpha)$.

Expected number of CIs that cover μ

If we construct n CIs each with probability $1-\alpha$ of including μ, then how many CIs do we expect will include μ. Since Y is the random number of CIs that will include the truth and $Y \sim Bin(n, 1-\alpha)$, we have

$$E[Y] = n(1-\alpha).$$

Calculate the expected number that will include the truth for the following scenarios:

- $n = 100, 1-\alpha = 0.95$ then $E[Y] = 100 \cdot 0.95 = 95$
- $n = 1000, 1-\alpha = 0.7$ then $E[Y] = 1000 \cdot 0.70 = 700$
- $n = 77, 1-\alpha = 0.66$ then $E[Y] = 77 \cdot 0.66 = 50.82$

If we are interested in how many will not cover the truth, this is the random variable $n - Y$ and $E[n-Y] = n - E[Y]$. Calculate the expected number that will not include the truth for the same scenarios:

- $n = 100, 1-\alpha = 0.95$ then $E[n-Y] = 100 - 95 = 5$
- $n = 1000, 1-\alpha = 0.7$ then $E[n-Y] = 1000 - 700 = 300$
- $n = 77, 1-\alpha = 0.66$ then $E[n-Y] = 77 - 50.82 = 26.18$

Summary

Here are the interpretation statements for a $100(1-\alpha)\%$ confidence interval for the population mean μ:

- Out of n $100(1-\alpha)\%$ confidence interval for μ, we expect $n(1-\alpha)$ confidence intervals to include/cover μ (and $n\alpha$ to not cover μ.
- We are $100(1-\alpha)\%$ confident that μ falls within the bounds of the constructed interval.

I really hate the second statement as I believe it gives you a false impression of what you have actually learned. The second interpretation DOES NOT tell you what you should believe, it is really a succinct version of the prevous interpretation.

When you see the words confidence or confident, think in your head, the word frequency.

Issues with a frequentist interpretation of probability

How can you interpret the following probability statements:

- What is the probability it will rain tomorrow?
- What is the probability the Vikings will their next game?
- What is the probability my unborn child has Down syndrome?
- What is the probability humans are the main cause of climate change on Earth?

Bayesian interpretation of probability

Interpretation

The Bayesian interpretation of probability is that probability is a statement about your degree of belief that an event will (or has) occurred.

Advantages:

- Can interpret probability for one time events.
- States what you should believe.
- Natural to make decisions based on your belief.
- Everyone has their own probability.

Disadvantages:

- Requires more math (integration).
- Requires you to specify your belief before seeing data.
- Has no relation to relative frequency.
- Everyone has their own probability.

Credible intervals

The Bayesian analog to confidence intervals are credible intevals. These intervals tell you where you should believe the parameter to be. Thus a $100(1-\alpha)\%$ credible interval for μ tells you that you should believe that the true population mean μ is in the interval with probability $1-\alpha$.

It turns out, under a particular prior, the confidence intervals that we construct are exactly the same as credible intervals. Thus, you will actually be correct even when you misinterpret confidence intervals.